

TOTALLY DISCONNECTED GROUPS FROM BAUMSLAG-SOLITAR GROUPS

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ABSTRACT. For each Baumslag-Solitar group $\text{BS}(m, n)$ ($m, n \in \mathbb{Z} \setminus \{0\}$), a totally disconnected, locally compact group, $G_{m,n}$, is constructed so that $\text{BS}(m, n)$ is identified with a dense subgroup of $G_{m,n}$. The scale function on $G_{m,n}$, a structural invariant for the topological group, is seen to distinguish the parameters m and n to the extent that the set of scale values is

$$\left\{ \left(\frac{\text{lcm}(m, n)}{|m|} \right)^\rho, \left(\frac{\text{lcm}(m, n)}{|n|} \right)^\rho \mid \rho \in \mathbb{N} \right\}.$$

It is also shown that $G_{m,n}$ has flat rank 1 when $|m| \neq |n|$ and 0 otherwise, and that $G_{m,n}$ has a compact, open subgroup isomorphic to the product $\{(\mathbb{Z}_p, +) \mid p \text{ is a prime divisor of the scale}\}$.

1. INTRODUCTION

In this article we investigate a family of totally disconnected, locally compact groups $G_{m,n}$ built from Baumslag-Solitar groups, which are the groups $\text{BS}(m, n)$ with presentations $\langle a, t \mid ta^m t^{-1} = a^n \rangle$ for non-zero integers m and n .

Recall that a topological group G is a group that is also a topological space such that the map $(x, y) \mapsto xy^{-1} : G \times G \rightarrow G$ is continuous. A topological space is *Hausdorff* if for any two points x, y there are disjoint open sets U, V , with $x \in U$ and $y \in V$, *totally disconnected* if the connected component containing any point is just the point itself, or equivalently that for any two points x, y there are disjoint open sets U, V , with $x \in U$ and $y \in V$ and whose union is the whole space, and *locally compact* if each point has a compact neighbourhood. The study of locally compact groups naturally splits into the study of connected and totally disconnected groups, since for every locally compact group G , the connected component G_0 of the identity is a closed normal subgroup, and G/G_0 is totally disconnected.

The subgroup $\langle a \rangle$ is *commensurated* by $\text{BS}(m, n)$, and we follow a construction described in [11, 16, 18, 20, 21] that uses a commensurated subgroup of an abstract group to construct a totally disconnected, locally compact group that is the completion of the abstract group with respect to a topology defined on it in terms of cosets of the commensurated subgroup. The group thus obtained from $\text{BS}(m, n)$ is denoted $G_{m,n}$.

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In [9] Gal and Januszkiewicz prove that Baumslag-Solitar groups are a-T-menable, by embedding them into the topological group of automorphisms of their corresponding *Bass-Serre* tree. The closure of $\text{BS}(m, n)$ in this group coincides with the group $G_{m,n}$ we construct.

A key invariant of a totally disconnected locally compact group is the set of positive integers corresponding to the *scales* of its elements, which we define in Section 2. The scale function and the related notion of minimizing subgroups were introduced by the second author [23, 24] and fill a similar role in the theory of totally disconnected groups to that played by eigenvalues and triangularising bases in linear algebra and the theory of matrix groups.

We prove that the set of scales for the group $G_{m,n}$ for $m, n \neq 0$ is

$$\left\{ \left(\frac{\text{lcm}(m, n)}{|m|} \right)^\rho, \left(\frac{\text{lcm}(m, n)}{|n|} \right)^\rho \mid \rho \in \mathbb{N} \right\}.$$

The scale thus partially distinguishes the parameters in the definition of $\text{BS}(m, n)$. In particular, for each pair of relatively prime integers $m, n > 0$ we obtain a distinct nondiscrete totally disconnected group. We offer two alternative proofs of this result: a combinatorial approach is given in Section 6; while in Section 8 the *tidying procedure* developed by the second author is used.

Another key invariant of totally disconnected locally compact groups is their *flat rank* (defined in Section 7). In [2] it is shown that the flat rank of a hyperbolic group (as a topological group) is at most 1. Here we show that $G_{m,n}$ has flat rank 1 for all $|m| \neq |n|$, even though none of these groups is hyperbolic.

The article is organised as follows. In Sections 2–4 we define scale, Baumslag-Solitar groups and commensurated subgroups, and list properties that will be used later. In Section 5 we outline the construction of a totally disconnected group starting with an abstract group having a commensurated subgroup, which we apply to Baumslag-Solitar groups. In Section 6 we compute the scale function for $G_{m,n}$. In Section 7 we compute a formula for the *modular function* for $G_{m,n}$, and compute its flat rank. In Section 8 we give more detail on the local structure of $G_{m,n}$, describing its compact open subgroups explicitly, and recompute the modular and scale functions with this description.

Note that throughout this paper \mathbb{N} will denote the set of nonnegative integers (including 0).

2. THE SCALE FUNCTION

Let G be a totally disconnected locally compact group. By van Dantzig's Theorem (see [22] or [12] Theorem 7.7), every neighbourhood of the identity contains a compact open subgroup, V say. An automorphism of G is a group automorphism $\alpha : G \rightarrow G$ that is also a topological homeomorphism, meaning α and α^{-1} are continuous. If α is an automorphism and V is a compact open subgroup, the set $\alpha^{-1}(V)$ is compact and open. The cosets of the subgroup $\alpha^{-1}(V) \cap V$ form an open cover of V , and since V is compact, the index $[V : \alpha^{-1}(V) \cap V]$ is finite. Define the *scale* of α , denoted $s(\alpha)$, to be the minimum such index over all compact open subgroups V of G . Then $s(\alpha)$ is the minimum of a set of positive integers and $s : \text{Aut}(G) \rightarrow \mathbb{Z}^+$ is a well defined function. A subgroup V for which $s(\alpha)$ is attained is called *minimizing* for α .

In the case that $\alpha = \alpha_x : g \mapsto xgx^{-1}$ is an inner automorphism, the scale function induces a function from the group to \mathbb{Z}^+ , also denoted by s , which enjoys the the following properties.

Proposition 2.1 ([23, 24]). *Let $s : G \rightarrow \mathbb{Z}^+$ be the scale function on G . Then*

- (i) s is continuous;
- (ii) for each $x \in G$ and $n \in \mathbb{N}$, $s(x^n) = s(x)^n$;
- (iii) if V is minimising for x then V is minimising for x^i for all $i \in \mathbb{Z}$;
- (iv) s is invariant under conjugation, that is, $s(x) = s(yxy^{-1})$ for any $x, y \in G$;
- (v) $s(x) = s(x^{-1}) = 1$ if and only if there is a compact open subgroup V with $x^{-1}Vx = V$.

Note that if G has the discrete topology then the scale of any element is 1, since $V = \{1\}$ is open and compact (and obviously for each $x \in G$, $x^{-1}\{1\}x = \{1\}$).

The continuity of the scale function implies the following.

Corollary 2.2. *If B is a dense subset of a totally disconnected locally compact group G , then $\{s(b) \mid b \in B\} = \{s(g) \mid g \in G\}$.*

Proof. If $g \in G$ has scale $s(g) = n$, then the inverse image U of the open set $\{n\}$ in \mathbb{Z}^+ under the scale function is an open set in G . Since B is dense in G , every open set contains points from B , so there is a point $b \in U \cap B$ with $s(b) = n$. \square

We will make use of an asymptotic formula of Möller which makes it possible to use arbitrary compact open subgroups to calculate the scale function.

Theorem 2.3 ([14] Theorem 7.7). *For any compact open subgroup V ,*

$$s(x) = \lim_{k \rightarrow \infty} [V : V \cap x^{-k}Vx^k]^{\frac{1}{k}}.$$

3. BAUMSLAG-SOLITAR GROUPS

In this section we collect some facts that will be useful in computing scales later on. Let $\text{BS}(m, n)$ be the group with presentation $\langle a, t \mid ta^m t^{-1} = a^n \rangle$ for non-zero integers m and n .

Define $\rho(w)$ to be the number of t letters minus the number of t^{-1} letters in the word $w \in \{a^{\pm 1}, t^{\pm 1}\}^*$. Recall Britton's lemma [13], which states that if a freely reduced nonempty word in the generators $a^{\pm 1}, t^{\pm 1}$ for $\text{BS}(m, n)$ is equal to the identity element, it must contain a subword of the form $ta^{cm}t^{-1}$ or $t^{-1}a^{cn}t$ for some $c \in \mathbb{Z}$ (such a subword is called a *pinch*). Replacing $ta^{cm}t^{-1}$ or $t^{-1}a^{cn}t$ by a^{cn} or a^{cm} in a word is called *removing a pinch*.

Lemma 3.1. *If $w, u \in \{a^{\pm 1}, t^{\pm 1}\}^*$ represent the same element in $\text{BS}(m, n)$ then $\rho(w) = \rho(u)$.*

Proof. If w or u contains a pinch, removing them does not change the respective values of ρ , so remove them (since each word has a finite number of t letters, this process will terminate). Since wu^{-1} equals the identity in $\text{BS}(m, n)$ it must contain a pinch by Britton's lemma. Assuming there are no pinches in w or u^{-1} , the pinch must consist of one t letter in one subword and one t^{-1} letter in the other. Removing all pinches in wu^{-1} until we obtain the empty word gives the result. \square

It follows that ρ is an invariant of group elements, and is called the *t-exponent sum* for w . The first author has made extensive use of the *t*-exponent sum to prove facts about Baumslag-Solitar groups [5, 6, 7, 8].

Lemma 3.2. *Each element in $\text{BS}(1, n)$ can be represented uniquely in the form $t^{-p}a^q t^r$ with $p, q, r \in \mathbb{Z}$, $p, r \geq 0$ and n dividing q only if $p = 0$ or $r = 0$.*

Proof. Let $w \in \{a^{\pm 1}, t^{\pm 1}\}^*$ be a word representing an element of $\text{BS}(1, n)$. Applying the moves $ta^{\pm 1} \rightarrow a^{\pm n}t$, $a^{\pm 1}t^{-1} \rightarrow t^{-1}a^{\pm n}$, and free cancellation, w is equal in the group to a word of the form $t^{-p}a^q t^r$ with $p, q, r \in \mathbb{Z}$, $p, r \geq 0$. If n divides q and $p, r > 0$ then the word contains a pinch $t^{-1}a^{nc}t$ which can be replaced by a^c . Repeating this gives the result. \square

Note that $\text{BS}(1, n)$ is isomorphic to the linear group, generated by

$$a = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad t = \begin{bmatrix} n & 0 \\ 0 & 1 \end{bmatrix}.$$

This group is isomorphic to

$$(1) \quad G = \left\{ \begin{bmatrix} n^\rho & z \\ 0 & 1 \end{bmatrix} \mid \rho \in \mathbb{Z}, z \in \mathbb{Z}[1/n] \right\}$$

and Lemma 3.2 thus represents a matrix in this group as

$$\begin{bmatrix} n^{-p} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & q \\ 0 & 1 \end{bmatrix} \begin{bmatrix} n^r & 0 \\ 0 & 1 \end{bmatrix}.$$

Lemma 3.3. *The subgroup $\langle a \rangle$ has a nontrivial subgroup that is normal in $\text{BS}(m, n)$ if and only if $|m| = |n|$.*

Proof. If $|m| = |n|$, then $\langle a^m \rangle$ is normal, in fact central, in $\text{BS}(m, n)$. If $|m| \neq |n|$, assume without loss of generality that $|m| < |n|$, and suppose $K \leq \langle a \rangle$ is a nontrivial normal subgroup of $\text{BS}(m, n)$. Then $K = \langle a^k \rangle$ for some positive integer k . Let s be the largest integer such that $k = q|n|^s + r$ for some $0 \leq q < |n|$ and $0 \leq r < |n|^s$. If $r = 0$ then $t^{-s}a^k t^s = a^{q(\pm m)^s}$ which is not in K since $|qm^s| < |qn^s| = k$, and if $r > 0$ then $t^{-s}a^k t^s$ is not in $\langle a \rangle$. \square

4. COMMENSURATED SUBGROUPS

Define a relation \sim on the set of subgroups of an abstract group G by $H \sim K$ if $H \cap K$ is finite index in both H and K . We say that H and K are *commensurable* if $H \sim K$. One may verify that being commensurable is an equivalence relation¹.

A subgroup H is *commensurated* by G if for each $g \in G$ the subgroups H and gHg^{-1} are commensurable. If $x, y \in G$, $xHx^{-1} \sim H$ and $H \sim yHy^{-1}$ then $xyHy^{-1}x^{-1} \sim H$ (since $H \sim K$ implies $xHx^{-1} \sim xKx^{-1}$). It follows that if G is generated by a set X , then H is commensurated by G if and only if $xHx^{-1} \sim H$ for all $x \in X$. This gives a fast way to check for commensurated subgroups in finitely generated groups.

One example of a commensurated subgroup is $\text{SL}(k, \mathbb{Z})$ in $\text{SL}(k, \mathbb{Q})$, and in this important case the construction about to be given in Section 5 yields the embedding of $\text{SL}(k, \mathbb{Q})$ into $\text{SL}(k, \mathbb{A}_f)$, where \mathbb{A}_f denotes the ring of finite adeles. Restricting to the subgroup G of $\text{SL}(2, \mathbb{Q})$ appearing in Equation (1), this construction yields an embedding of G into a group isomorphic to

$$(2) \quad \left\{ \begin{bmatrix} n^\rho & z \\ 0 & 1 \end{bmatrix} \mid \rho \in \mathbb{Z}, z \in \mathbb{Q}_{p_1} \times \cdots \times \mathbb{Q}_{p_l} \right\},$$

¹ Clearly \sim is reflexive and symmetric. That \sim is transitive follows because, for subgroups H , K and L , $[H : H \cap L] \leq [H : H \cap K \cap L] = [H : H \cap K][H \cap K : H \cap K \cap L]$ and $[H \cap K : H \cap K \cap L] \leq [K : K \cap L]$.

where p_1, \dots, p_l are the prime divisors of n . In the case of Baumslag-Solitar groups, the cyclic subgroup $\langle a \rangle$ is commensurated because $t\langle a \rangle t^{-1} \cap \langle a \rangle = \langle a^n \rangle$ is finite index in both $t\langle a \rangle t^{-1}$ and $\langle a \rangle$, and similarly $t^{-1}\langle a \rangle t \cap \langle a \rangle = \langle a^m \rangle$ is finite index in both $t^{-1}\langle a \rangle t$ and $\langle a \rangle$. Recalling the isomorphism between $\text{BS}(1, n)$ and G , applying the construction to $\text{BS}(m, n)$ will thus extend this embedding of matrix groups.

5. CONSTRUCTION OF THE TOTALLY DISCONNECTED GROUP

Let G be an abstract group with commensurated subgroup H . The action of G on G/H given by $g'.gH = (g'g)H$, ($g, g' \in G$), determines a homomorphism $\pi : G \rightarrow \text{Sym}(G/H)$. For each $x \in \text{Sym}(G/H)$ and each finite subset F of G/H , define

$$\mathcal{N}(x, F) = \{y \in \text{Sym}(G/H) \mid y(gH) = x(gH) \ \forall (gH) \in F\}.$$

Note that if $y \in \mathcal{N}(x, F)$ then $\mathcal{N}(x, F) = \mathcal{N}(y, F)$. If $\mathcal{N}(x_1, F_1) \cap \mathcal{N}(x_2, F_2)$ is nonempty then it contains some element y , so $\mathcal{N}(x_1, F_1) = \mathcal{N}(y, F_1)$ and $\mathcal{N}(x_2, F_2) = \mathcal{N}(y, F_2)$. Then

$$\mathcal{N}(x_1, F_1) \cap \mathcal{N}(x_2, F_2) = \mathcal{N}(y, F_1) \cap \mathcal{N}(y, F_2) = \mathcal{N}(y, F_1 \cup F_2).$$

It follows that $\{\mathcal{N}(x, F) \mid x \in G, F \subseteq G/H \text{ finite}\}$ forms a basis for a topology on $\text{Sym}(G/H)$.

Lemma 5.1. *The topology defined on $\text{Sym}(G/H)$ is Hausdorff. Hence the subspace topology on $\pi(G)$ induces a Hausdorff topology on $G/\ker \pi$.*

Proof. If $x, y \in \text{Sym}(G/H)$ are distinct, then the neighbourhoods $\mathcal{N}(x, \{gH\})$ and $\mathcal{N}(y, \{gH\})$ are disjoint for some $g \in G$. Hence $\text{Sym}(G/H)$ is a Hausdorff topological group. The second claim is justified by the First Isomorphism Theorem. \square

Note that $\ker \pi$ is a subgroup of H and that $H/\ker \pi$ is commensurated by $G/\ker \pi$. The topology on $G/\ker \pi$ may then be defined equivalently by considering the injection of $G/\ker \pi$ into $\text{Sym}(G/\ker \pi)/(H/\ker \pi)$. From now on it is assumed that the kernel is trivial.

Lemma 5.2. *The topology defined is totally disconnected.*

Proof. If $x, y \in \text{Sym}(G/H)$ are distinct, there is a coset gH with $x(gH) \neq y(gH)$. Then $\mathcal{N}(x, \{gH\})$ is an open set containing x , and its complement,

$$\bigcup \{\mathcal{N}(z, \{gH\}) \mid z(gH) \neq x(gH)\},$$

is open and contains y . \square

It is a standard result that $\text{Sym}(G/H)$ is a topological group with the topology defined. For completeness we include a proof of this fact.

Lemma 5.3. *The map $* : (x, y) \mapsto xy^{-1}$ is continuous.*

Proof. The map is continuous at (x, y) if for any open set V containing xy^{-1} there is an open set $U \subseteq \text{Sym}(G/H) \times \text{Sym}(G/H)$ containing (x, y) whose image is in V .

Since V is open and contains xy^{-1} , it contains a set $\mathcal{N}(xy^{-1}, F)$ for some finite F . Take $U = \mathcal{N}(x, y^{-1}F) \times \mathcal{N}(y, y^{-1}F)$. If $(a, b) \in U$, then $b(y^{-1}(gH)) = y(y^{-1}(gH)) = (gH)$ and $b^{-1}(gH) = y^{-1}(gH)$ for all $gH \in F$ and $ab^{-1}(gH) = a(y^{-1}(gH)) = xy^{-1}(gH)$ for all $gH \in F$. Hence $ab^{-1} \in \mathcal{N}(xy, F) \subseteq V$. \square

It follows that $\pi(G)$ is also a (Hausdorff and totally disconnected) topological group, with the subspace topology induced from $\text{Sym}(G/H)$. Define $G//H$ to be the closure of $\pi(G)$ in $\text{Sym}(G/H)$, and define \tilde{H} to be the closure of $\pi(H)$. Then

$$\tilde{H} = \{x \in G//H \mid x(H) = H\} = G//H \cap \mathcal{N}(e, \{H\})$$

is an open subgroup of $G//H$.

It is shown next that $G//H$ is locally compact. The following fact is used in this argument and also later in the calculation of the scale.

Lemma 5.4. *Suppose that V is an open subgroup of $G//H$. Then, setting $U = V \cap \pi(G)$, each U -orbit $U.(gH) \subseteq G/H$ is equal to V -orbit $V.(gH)$.*

Proof. Since V is open and $\pi(G)$ is dense in $G//H$, U is a dense subgroup of V . Hence the intersection $\mathcal{N}(x, \{gH\}) \cap U$ is nonempty for any $x \in V$ and there is $u \in U$ such that $x(gH) = u(gH)$. \square

When applied to the orbits $U.xH$ and $V.xH$, the Orbit-Stabiliser theorem implies the following consequence.

Corollary 5.5. *If V and U are as in Lemma 5.4 and $x \in G//H$, then*

$$[V : x^{-1}Vx \cap V] = [U : x^{-1}Ux \cap U].$$

Lemma 5.6. *The subspace topology defined on $G//H$ is locally compact.*

Proof. Lemma 5.4 implies that H and \tilde{H} act on G/H by permuting cosets in the blocks $H(gH)$ ($g \in G$) so that, making a natural identification,

$$(3) \quad \tilde{H} \leq \prod_{H(gH) \subseteq G/H} \text{Sym}(H(gH)).$$

It may be checked that \tilde{H} is a closed subgroup under this identification and that the topology on \tilde{H} is equivalent to the subspace topology for the product topology on $\prod_{H(gH) \subseteq G/H} \text{Sym}(H(gH))$. Since commensurability of H implies that each block $H(gH)$ is finite, it follows by Tychonov's theorem (see for example [15] Theorem 6.50) that \tilde{H} is compact, and is thus a compact, open neighbourhood of e . \square

The subgroup $\langle a \rangle$ is commensurated by the group $\text{BS}(m, n)$ for $m, n \neq 0$ and the kernel of the map $\pi : \text{BS}(m, n) \rightarrow \text{Sym}(\text{BS}(m, n)/\langle a \rangle)$ is a subgroup of $\langle a \rangle$. Then $\ker \pi$ is trivial if $|m| \neq |n|$, by Lemma 3.3, and $\text{BS}(m, n)$ embeds as a dense subgroup of the totally disconnected, locally compact group $\text{BS}(m, n)//\langle a \rangle$. This group is denoted by $G_{m,n}$ in the sequel.

If $|m| = |n|$, we have $a^i(gH) = gH$ for all $g \in \text{BS}(m, \pm m)$ if and only if m divides i and so $\ker(\pi) = \langle a^m \rangle$. Factoring $\text{BS}(m, \pm m)$ by $\ker(\pi)$ then makes the commensurated subgroup $\langle a \rangle/\langle a^m \rangle$ finite, whence \tilde{H} is finite and discrete. Hence

$$(\text{BS}(m, \pm m)/\langle a^m \rangle)//(\langle a \rangle/\langle a^m \rangle) \cong \text{BS}(m, \pm m)/\langle a^m \rangle$$

(which we will denote as $G_{m, \pm m}$) and is discrete. Since the construction yields nothing new in this case, the focus of the rest of the article is on the case $|m| \neq |n|$.

5.1. An alternative construction of $G_{m,n}$. Gal and Januszkiewicz [9] embed $\text{BS}(m,n)$ into a topological group as follows. Recall the *Bass-Serre tree* for a graph of groups [1, 19]. In the case of $\text{BS}(m,n)$ the tree has a vertex for each coset of $\langle a \rangle$, and $(u\langle a \rangle, v\langle a \rangle)$ is a directed edge labeled t^ϵ if $v\langle a \rangle = ut^\epsilon\langle a \rangle$ where $\epsilon = \pm 1$. As an example, part of the Bass-Serre tree for the group $\text{BS}(2,3)$ is shown in Figure 1. Note that each vertex will have degree 5 in this example (two outgoing edges labeled t and three labeled t^{-1}).

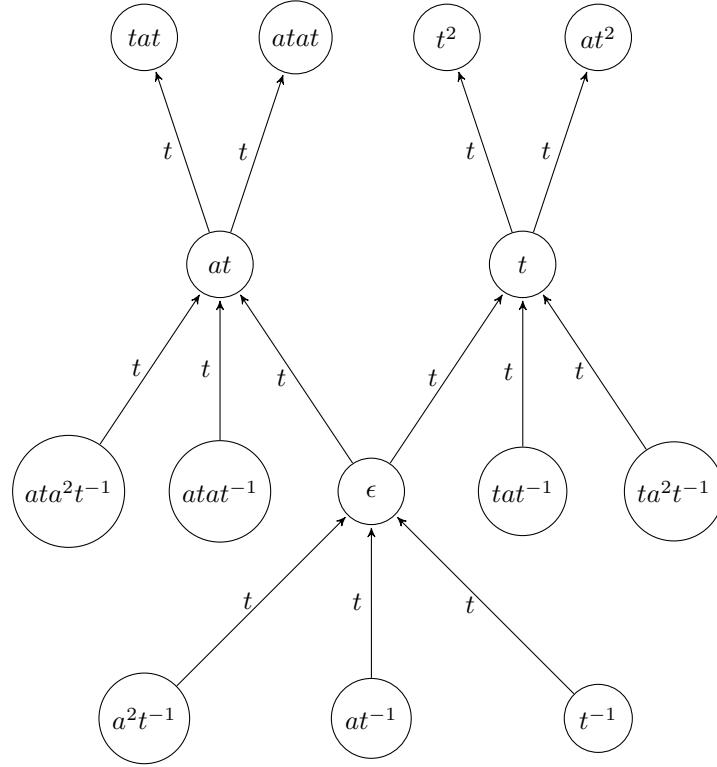


FIGURE 1. Part of the Bass-Serre tree for $\text{BS}(2,3)$. Edges labeled t^{-1} correspond to traveling in reverse direction along edges labeled t .

Let $\text{Aut}(T_{m,n})$ be the group of automorphisms of the Bass-Serre tree for $\text{BS}(m,n)$ with vertex set $V(T_{m,n})$, and let $\{N(x, \mathcal{F}) \mid x \in \text{Aut}(T_{m,n}), \mathcal{F} \subset^{\text{finite}} V(T_{m,n})\}$ be a base of neighborhoods for a topology on $\text{Aut}(m,n)$, where

$$N(x, \mathcal{F}) = \{\beta \in \text{Aut}(X) \mid x.v = \beta.v \text{ for all } v \in \mathcal{F}\}.$$

Since vertices correspond to (left) cosets of $\langle a \rangle$ in $\text{BS}(m,n)$ then $\text{Aut}(T_{m,n}) \subseteq \text{Sym}(\text{BS}(m,n)/\langle a \rangle)$.

To see that the closure of $\text{BS}(m,n)$ in $\text{Aut}(T_{m,n})$ coincides with $G_{m,n}$ defined above, if $x \in G_{m,n}$ then either $x \in \text{BS}(m,n)$ or every open set in $\text{Sym}(\text{BS}(m,n)/\langle a \rangle)$

containing x contains an element of $\text{BS}(m, n)$. So for every finite set of cosets (vertices of $T_{m,n}$) x agrees with some element of $\text{BS}(m, n)$ and so preserves adjacencies of the Bass-Serre tree.

6. COMPUTING SCALES FOR $G_{m,n}$

In this section the scales of elements of $G_{m,n}$ are computed when $|m| \neq |n|$. For convenience we will abuse notation and identify elements and subsets of $\text{BS}(m, n)$ with their images under the embedding π .

Since $\text{BS}(m, n)$ is dense in $G_{m,n}$, Corollary 2.2 shows that in order to compute scales in $G_{m,n}$, it suffices to compute the scale of elements in $\text{BS}(m, n)$. If V is a compact open subgroup of $G_{m,n}$ then Corollary 5.5 shows that the index $[V : x^{-1}Vx \cap V]$ is the same as the index $[U : x^{-1}Ux \cap U]$ where U is the intersection of V with $\text{BS}(m, n)$. It follows that scales in $G_{m,n}$ can be computed by working entirely in $\text{BS}(m, n)$.

We start by considering the case when one of m, n is a proper divisor of the other. Note that this includes the case that one of m, n is ± 1 , in which case the scale could be computed more easily by working directly with the topological matrix group in Equation (2), or with its dense subgroup described in Equation (1).

Suppose that $n = mr$ with $|r| > 1$. Then

$$\begin{aligned} t^{-1}\langle a^m \rangle t \cap \langle a^m \rangle &= \langle a^m \rangle, \\ t^{-1}\langle a^{mr^j} \rangle t \cap \langle a^m \rangle &= \langle a^{mr^{j-1}} \rangle, \quad \text{and} \\ t\langle a^{mr^i} \rangle t^{-1} \cap \langle a^m \rangle &= \langle a^{mr^{i+1}} \rangle \end{aligned}$$

for any $i \geq 0, j > 0$. These facts may be encoded in a graph. Define Λ to be the labeled directed graph having nodes $N = \{mr^i \mid i \in \mathbb{N}\}$, and directed edges

$$E = \{(m, m), (mr^j, mr^{j-1}), (mr^i, mr^{i+1}) \mid j > 0, i \geq 0\},$$

where the loop (m, m) and edges (mr^j, mr^{j-1}) are labeled t , and edges (mr^i, mr^{i+1}) are labeled t^{-1} . Then (x, y) is an edge labeled t^ϵ if and only if

$$t^{-\epsilon}\langle a^x \rangle t^\epsilon \cap \langle a^y \rangle = \langle a^y \rangle.$$

A picture of part of Λ is shown in Figure 2.

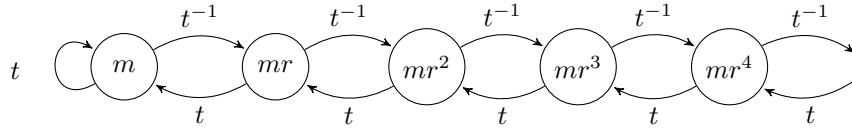


FIGURE 2. Part of the graph Λ . An edge from x to y labeled t^ϵ means that $t^{-\epsilon}\langle a^x \rangle t^\epsilon \cap \langle a^y \rangle = \langle a^y \rangle$.

Recall Britton's lemma and the notion of a pinch from Section 3.

Lemma 6.1. *Let $u = a^\eta t^\epsilon v$ with $\eta \in \mathbb{Z}$ and $\epsilon = \pm 1$ be a freely reduced word in $\text{BS}(m, n)$ with no pinches. Then*

$$u^{-1}\langle a^i \rangle u \cap \langle a \rangle = v^{-1}(t^{-\epsilon}\langle a^i \rangle t^\epsilon \cap \langle a \rangle) v \cap \langle a \rangle.$$

Proof. If the word $u^{-1}(a^i)^j u = v^{-1}t^{-\epsilon}a^{-\eta}(a^i)^j a^\eta t^\epsilon v$ is in $\langle a \rangle$ then it must contain a pinch. There is no pinch within $t^\epsilon v$ (or $v^{-1}t^{-\epsilon}$) so the pinch must be $t^{-\epsilon}(a^i)^j t^\epsilon$, so this subword must be in $\langle a \rangle$. \square

Lemma 6.2. *Let w be a freely reduced word with no pinches, and let $p(w)$ be the word in the free monoid generated by t and t^{-1} obtained by removing all $a^{\pm 1}$ letters from w . Consider the path in Λ starting at the node mr^i , whose edge labels follow the sequence $p(w)$. Then*

$$w^{-1}\langle a^{mr^i} \rangle w \cap \langle a^m \rangle = \langle a^{mr^k} \rangle$$

where mr^k is the label of the node at the end of this path.

Proof. If w has no $t^{\pm 1}$ letters then the statement is clearly true. Suppose for induction the statement is true for words with $q t^{\pm 1}$ letters. Let u be a freely reduced word with no pinches with $q+1$ $t^{\pm 1}$ letters, and write $u = a^\eta t^\epsilon v$, where $\epsilon = \pm 1$ and $\eta \in \mathbb{Z}$. Then by Lemma 6.1 we have

$$\begin{aligned} u^{-1}\langle a^{mr^i} \rangle u \cap \langle a^m \rangle &= \left(v^{-1}t^{-\epsilon}\langle a^{mr^i} \rangle t^\epsilon v \cap \langle a \rangle \right) \cap \langle a^m \rangle \\ &= \left(v^{-1}\left(t^{-\epsilon}\langle a^{mr^i} \rangle t^\epsilon \cap \langle a \rangle \right) v \cap \langle a \rangle \right) \cap \langle a^m \rangle \\ &= \left(v^{-1}\langle a^{mr^j} \rangle v \cap \langle a \rangle \right) \cap \langle a^m \rangle \end{aligned}$$

where (mr^i, mr^j) is a directed edge labeled t^ϵ in Λ . The path starting at mr^i labeled by $p(u)$ consists of this edge to mr^j , followed by a path labeled $p(v)$. By inductive assumption

$$\left(v^{-1}\langle a^{mr^j} \rangle v \cap \langle a \rangle \right) \cap \langle a^m \rangle = v^{-1}\langle a^{mr^j} \rangle v \cap \langle a^m \rangle = \langle a^{mr^k} \rangle$$

since mr^k is the endpoint of the path $p(v)$ starting at mr^j . The result follows. \square

Consider the following example which shows how the previous lemma can be used to compute the scale. Let $w = t^4at^{-2}a$. Then by tracing the path $p(w) = t^4t^{-2}$ starting at m through Λ we have $w^{-1}\langle a^m \rangle w \cap \langle a^m \rangle = \langle a^{mr^2} \rangle$. Since scale is invariant under conjugation, we could instead consider $u = t^{-2}at^4a$, in which case the path $p(u) = t^{-2}t^4$ starting at m ends at m , so $u^{-1}\langle a^m \rangle u \cap \langle a^m \rangle = \langle a^m \rangle$, and we see that (the closure of) $\langle a^m \rangle$ is minimising for u , and so $s(w) = s(u) = 1$.

This pre-conjugating step is the key to proving Proposition 6.4 below. We will need the following fact.

Lemma 6.3. *Let $x \in \text{BS}(m, n)$. Then x is conjugate to a word $w \in \{a^{\pm 1}, t^{\pm 1}\}^*$ such that ww is freely reduced and contains no pinches.*

Proof. Let $y \in \{a^{\pm 1}, t^{\pm 1}\}^*$ be a word equal to the element x in $\text{BS}(m, n)$, and consider the following four moves:

- (1) if y is not freely reduced, then removing a cancelling pair reduces the length of y , and gives a word z equal to y and shorter in length.
- (2) if y is freely reduced and yy is not freely reduced, then we must have $y = c^{\pm 1}zc^{\mp 1}$, so z is conjugate to y and shorter in length.
- (3) if y contains a pinch, then removing gives a word z equal to y with fewer t letters.

(4) if y has no pinches and yy contains a pinch, then we must have $y = a^i t^{-1} v t a^j$ with $i + j = km$, or $y = a^i t v t^{-1} a^j$ with $i + j = kn$. Conjugating by $a^i t$ [or $a^i t^{-1}$ respectively] then removing the pinch we obtain a word $z = v a^{nk}$ [or $z = v a^{mk}$] conjugate to y with fewer t letters.

Define an order \prec on words $y, z \in \{a^{\pm 1}, t^{\pm 1}\}^*$ by $z \prec y$ if y has more t letters than z , or they have the same number of t letters but y is longer than z . Then since each of the four moves produces a word that is shorter in this order, after a finite number of moves the word w with the required properties is obtained. \square

Proposition 6.4. *If $x \in \text{BS}(m, mr)$ for $|r| > 1$ is equal to a word of t -exponent sum ρ , then $s(x) = 1$ if $\rho \geq 0$ and $s(x) = |r|^{\rho|}$ if $\rho < 0$.*

Proof. Suppose x has nonnegative t -exponent sum ρ and, applying Lemma 6.3, choose w conjugate to x and with ww freely reduced and containing no pinches. We consider two cases.

If w contains no t^{-1} letters, then the path starting at m labeled by $p(w) = t^\rho$ ends at m , so Lemma 6.2 we have $w^{-1} \langle a^m \rangle w \cap \langle a^m \rangle = \langle a^m \rangle$ and $s(w) = s(x) = 1$.

Otherwise w contains a t^{-1} letter, and since $\rho \geq 0$ it is cyclically conjugate to a word $u = t^{-1} w_1 t a^\eta$, which is freely reduced and contains no pinches since it is a subword of ww . Suppose u contains k t letters, and consider the word $v = t^{-k} u t^k$. Note that $k > \rho$. This word is freely reduced and contains no pinches since u starts with t^{-1} and ends with ta^η .

Now consider the path in Λ starting at m labeled $p(v)$. It first travels k steps to the right, then since $p(u)$ has exactly k t letters, it travels left for each t and right for each t^{-1} in $p(u)$ (that is, we have ensured that it never enters the loop at m). Since the subpath labeled $p(u)$ started at mr^k and the t -exponent sum of u is $\rho \geq 0$, the subpath ends at $mr^{k-\rho}$. Finally the path $p(v)$ travels $k - \rho$ edges left and the ρ edges around the loop at m , and so $u^{-1} \langle a^m \rangle u \cap \langle a^m \rangle = \langle a^m \rangle$ and $s(u) = s(w) = s(x) = 1$.

To compute the scale of x of negative t -exponent ρ , choose w conjugate to x and with ww freely reduced and containing no pinches. If w contains no t letters then w^{-1} contains no t^{-1} letters, so by the argument above the closure of $\langle a^m \rangle$ is minimising for w^{-1} , so is minimising for w by Proposition 2.1(iii). Applying Lemma 6.2, the path starting at m labeled by $p(w) = t^\rho$ ends at $mr^{|\rho|}$ so $w^{-1} \langle a^m \rangle w \cap \langle a^m \rangle = \langle a^{mr^{|\rho|}} \rangle$ which has index $|r|^{\rho|}$ in $\langle a^m \rangle$, so $s(w) = s(x) = |r|^{\rho|}$.

If w contains t letters, then w is cyclically conjugate to $u = t^{-1} u_1 t a^\eta$, and so conjugate to $v = t^{-k} u t^k$, where k is the number of t letters in u , and u, v are freely reduced and contain no pinches.

Since $v^{-1} = t^{-k} (a^{-\eta} t^1 u_1^{-1} t) t^k$, the argument for the second case above shows that $\langle a^m \rangle$ is minimising for v^{-1} , so it is minimising for v . Using Lemma 6.2 once again, the path in Λ starting at m labeled $p(v)$ travels k steps to the right, then since $p(u)$ has exactly k t letters, it travels left for each t and right for each t^{-1} in $p(u)$ (once again, we have ensured that it never enters the loop at m).

Since the subpath labeled $p(u)$ started at mr^k and never enters the loop, and the t -exponent sum of u is $\rho < 0$, the subpath ends at $mr^{k+|\rho|}$. Finally the path $p(v)$ travels k edges left to end at $mr^{|\rho|}$, and $s(v^{-1} \langle a^m \rangle v \cap \langle a^m \rangle = \langle a^{mr^{|\rho|}} \rangle$ which has index $|r|^{\rho|}$ in $\langle a^m \rangle$ so $s(v) = s(x) = |r|^{\rho|}$. \square

We now turn to the case where neither m or n is a divisor of the other. In this case we make use of Möller's formula (Theorem 2.3), choosing V to be closure of $\langle a \rangle$. For this, it is necessary to compute $w^{-k}\langle a \rangle w^k \cap \langle a \rangle$ for arbitrary natural numbers k .

Corollary 6.5. *Let $x \in \text{BS}(m, n)$. Then x is conjugate to a word $w \in \{a^{\pm 1}, t^{\pm 1}\}^*$ such that w^k is freely reduced and contains no pinches for all $k \in \mathbb{N}$.*

Proof. By Lemma 6.3, x is conjugate to w such that ww is freely reduced and contains no pinches. If w^k contains a pair yy^{-1} for some $y \in \{a^{\pm 1}, t^{\pm 1}\}$, then yy^{-1} must lie either in w or ww . If w^k contains a pinch, and the pinch is of the form $t^\epsilon uw^i vt^{-\epsilon}$ where v, u are a prefix and suffix of w , then w^i must consist of $a^{\pm 1}$ letters, meaning $i = 0$, so ww contains the pinch. \square

Define a directed graph Ω as follows. Let $l = \text{lcm}(m, n)$. The nodes of Ω are labeled by integers from the set

$$N = \left\{ 1, m \left(\frac{l}{n} \right)^i, l \left(\frac{l}{n} \right)^i, n \left(\frac{l}{m} \right)^i, l \left(\frac{l}{m} \right)^i \mid i \in \mathbb{N} \right\}.$$

These integers are distinct because neither m nor n divides the other.

We define directed labeled edges in Ω in a similar way to edges in Λ : (x, y) is an edge from x to y labeled t^ϵ for $\epsilon = \pm 1$ if $t^{-\epsilon}\langle a^x \rangle t^\epsilon \cap \langle a \rangle = \langle a^y \rangle$. Note that this time we intersect with $\langle a \rangle$ rather than $\langle a^m \rangle$.

Lemma 6.6. *The edge set E of Ω consists of the following edges, for all positive integers i, j :*

$$\begin{array}{ll} (1, m), & (1, n), \\ (m, m \left(\frac{l}{n} \right)), & (n, n \left(\frac{l}{m} \right)), \\ (m \left(\frac{l}{n} \right)^i, m \left(\frac{l}{n} \right)^{i+1}), & (n \left(\frac{l}{m} \right)^i, n \left(\frac{l}{m} \right)^{i+1}), \end{array}$$

and

$$\begin{array}{ll} (n, m), & (m, n), \\ (n \left(\frac{l}{m} \right)^i, l \left(\frac{l}{m} \right)^{i-1}), & (m \left(\frac{l}{n} \right)^i, l \left(\frac{l}{n} \right)^{i-1}), \\ (l \left(\frac{l}{n} \right)^i, m \left(\frac{l}{n} \right)^{i+1}), & (l \left(\frac{l}{m} \right)^i, n \left(\frac{l}{m} \right)^{i+1}), \\ (l \left(\frac{l}{n} \right)^i \left(\frac{l}{m} \right)^j, l \left(\frac{l}{n} \right)^{i-1} \left(\frac{l}{m} \right)^{j+1}), & (l \left(\frac{l}{n} \right)^i \left(\frac{l}{m} \right)^j, l \left(\frac{l}{n} \right)^{i+1} \left(\frac{l}{m} \right)^{j-1}) \end{array}$$

where all edges in the left column are labeled t , and all edges in the right column are labeled t^{-1} .

Proof. Since $1 \in N$ and

$$\begin{aligned} t^{-1}\langle a \rangle t \cap \langle a \rangle &= \langle a^m \rangle \\ t\langle a \rangle t^{-1} \cap \langle a \rangle &= \langle a^n \rangle \end{aligned}$$

we have $m, n \in N$ and $(1, m), (1, n) \in E$ with $(1, m)$ labeled t and $(1, n)$ labeled t^{-1} .

Since

$$\begin{aligned}
t^{-1}\langle a^m \rangle t &= \{t^{-1}a^{|m|c}t : c \in \mathbb{Z}\} \\
&= \{t^{-1}a^{|m|(q\frac{l}{|m|}+r)}t : q, r \in \mathbb{Z}, 0 \leq r < \frac{l}{|m|}\} \\
&= \{t^{-1}a^{ql}tt^{-1}a^{|m|r}t : q, r \in \mathbb{Z}, 0 \leq r < \frac{l}{|m|}\} \\
&= \{a^{q\frac{l}{n}m}t^{-1}a^{|m|r}t : q, r \in \mathbb{Z}, 0 \leq r < \frac{l}{|m|}\}
\end{aligned}$$

has intersection $\langle a^{\frac{l}{n}m} \rangle$ with $\langle a \rangle$, we have the edge $(m, m(\frac{l}{n}))$ labeled t . A similar argument gives the edge $(n, n(\frac{l}{m}))$ labeled t^{-1} .

More generally for any $i \in \mathbb{N}$ we have

$$\begin{aligned}
t^{-1}\langle a^{m(\frac{l}{n})^i} \rangle t &= \{t^{-1}a^{|m|(\frac{l}{n})^i c}t : c \in \mathbb{Z}\} \\
&= \{t^{-1}a^{|m|(\frac{l}{n})^i(q\frac{l}{|m|}+r)}t : q, r \in \mathbb{Z}, 0 \leq r < \frac{l}{|m|}\} \\
&= \{t^{-1}a^{ql(\frac{l}{n})^i}tt^{-1}a^{|m|(\frac{l}{n})^i r}t : q, r \in \mathbb{Z}, 0 \leq r < \frac{l}{|m|}\} \\
&= \{a^{q\frac{l}{n}(\frac{l}{n})^i m}t^{-1}a^{|m|(\frac{l}{n})^i r}t : q, r \in \mathbb{Z}, 0 \leq r < \frac{l}{|m|}\}
\end{aligned}$$

which has intersection $\langle a^{n(\frac{l}{m})^i m} \rangle = \langle a^{(\frac{l}{n})^{i+1} m} \rangle$ with $\langle a \rangle$, so we have edge

$$\left(m\left(\frac{l}{n}\right)^i, m\left(\frac{l}{n}\right)^{i+1} \right)$$

labeled t for all $i \in \mathbb{N}$. A similar argument gives the edges

$$\left(n\left(\frac{l}{m}\right)^i, n\left(\frac{l}{m}\right)^{i+1} \right)$$

labeled t^{-1} .

In Figure 3 we have drawn part of the graph Ω , with these edges drawn vertically down the left and right sides of the picture.

The remaining edges (which are drawn horizontally in the picture) all follow from these facts where $c \in \mathbb{N}$:

$$\begin{aligned}
t\langle a^{mc} \rangle t^{-1} &= \langle a^{nc} \rangle, \\
t^{-1}\langle a^{lc} \rangle t &= \langle a^{n\frac{l}{m}c} \rangle, \\
t^{-1}\langle a^{nc} \rangle t &= \langle a^{mc} \rangle, \\
t^{-1}\langle a^{lc} \rangle t &= \langle a^{m\frac{l}{n}c} \rangle.
\end{aligned}$$

Since we have described an edge labeled t and t^{-1} starting from each node in the set N , there are no edges other than these. \square

Corollary 6.7. *The graph Ω is connected. The length of the shortest directed path from $m(\frac{l}{n})^i$ to $n(\frac{l}{m})^i$ is $i+1$ for all $i \in \mathbb{N}$, and also $i+1$ from $n(\frac{l}{m})^i$ to $m(\frac{l}{n})^i$.*

Proof. Each node in N can be reached by a path starting at 1 and labeled $t^i t^{-j}$ for $i, j \in \mathbb{N}$, $i \geq j$, as follows.

- The path t^i starting at 1 ends at $m(\frac{l}{n})^{i-1}$, for $i \geq 1$.
- The path $t^i t^{-j}$ starting at 1 ends at $l(\frac{l}{m})^{j-1}(\frac{l}{n})^{i-j-1}$, for $i \geq 2, j \geq 1$, and $j < i$.
- The path $t^i t^{-i}$ starting at 1 ends at $n(\frac{l}{m})^{i-1}$, for $i \geq 1$.

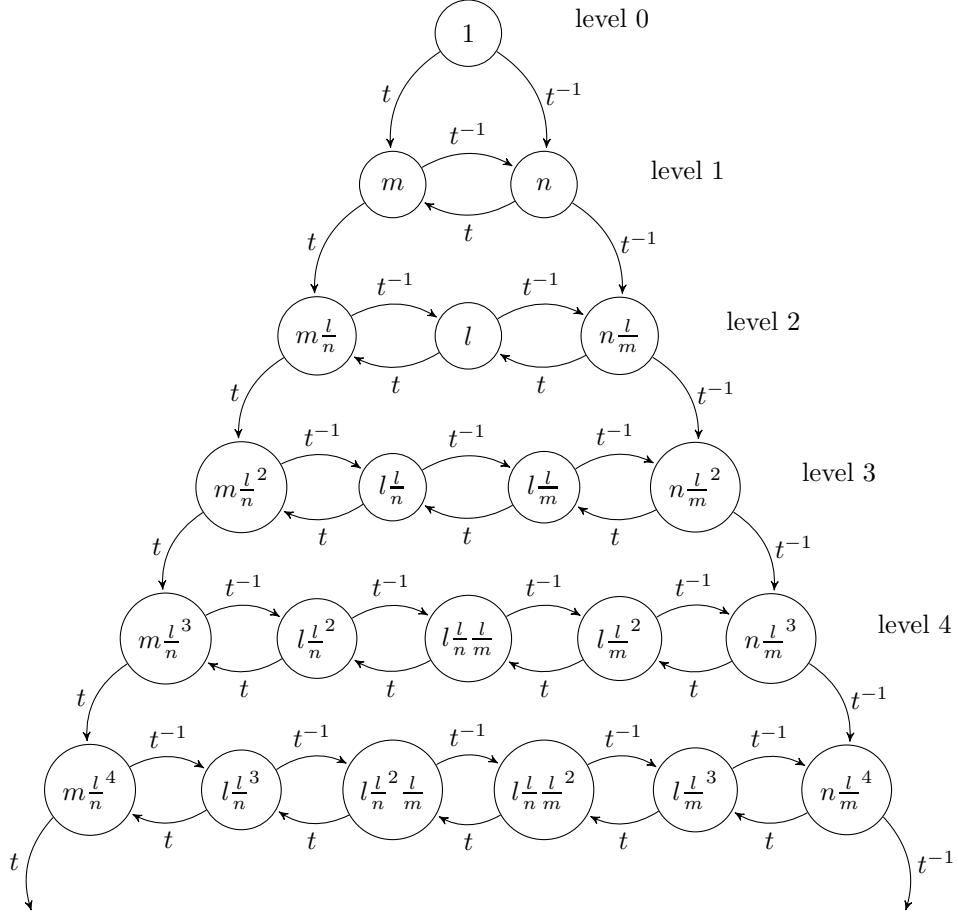


FIGURE 3. Part of the graph Ω . An edge from x to y labeled t^ϵ means that $t^{-\epsilon}\langle a^x \rangle t^\epsilon \cap \langle a \rangle = \langle a^y \rangle$.

The path t^{-i} starting at $m \left(\frac{l}{n}\right)^{i-1}$ ends at $n \left(\frac{l}{m}\right)^{i-1}$ and similarly for the reverse path t^i starting at $n \left(\frac{l}{m}\right)^{i-1}$, so the distance from one side of Ω to the other at level i is i . \square

Lemma 6.8. *Let $w \in \text{BS}(m, n)$ be freely reduced and contain no pinches. Let $p(w)$ be the word in the free monoid generated by t and t^{-1} obtained from w by removing all $a^{\pm 1}$ letters. Consider the path starting at the node 1 in Ω and following the sequence of edges labeled by $p(w)$. Let r be the label of the node at the end of this path. Then $w^{-1}\langle a \rangle w \cap \langle a \rangle = \langle a^r \rangle$.*

Proof. The result follows from Lemma 6.1 and induction on the number of $t^{\pm 1}$ letters in w . If $w = a^{\eta_0}ta^{\eta_1}, \eta_0, \eta_1 \in \mathbb{Z}$ then $w^{-1}\langle a \rangle w \cap \langle a \rangle = a^{-\eta_1}t^{-1}\langle a \rangle ta^{\eta_1} \cap \langle a \rangle = \langle a^m \rangle$ and the path from 1 labeled t ends at m , with a similar result for $w = a^{\eta_0}t^{-1}a^{\eta_1}$.

If $w = uta^\eta$, $\eta \in \mathbb{Z}$, say the path $p(u)$ ends at the node r . Then $w^{-1}\langle a \rangle w \cap \langle a \rangle = a^{-\eta}t^{-1}u^{-1}\langle a \rangle uta^\eta \cap \langle a \rangle = a^{-\eta}t^{-1}\langle a^r \rangle ta^\eta \cap \langle a \rangle = \langle a^s \rangle$ where (r, s) is an edge labeled t in Ω . A similar argument applies for $w = ut^{-1}a^\eta$. \square

Lemma 6.9. *Let $w \in \{a^{\pm 1}, t^{\pm 1}\}^*$ with t -exponent sum $\rho(w)$, and let $t_{\max}(w)$ be the maximum t -exponent sum of any prefix of w . Write $w = w_1w_2$ where w_1 has t -exponent sum $t_{\max}(w)$, and w_2 has t -exponent sum $\mu(w) = \rho(w) - t_{\max}(w) \leq 0$. In other words, w_1 is the longest prefix of w that has maximum t -exponent sum, and the t -exponent sum of w_2 is nonpositive. Let r be the number of t^{-1} letters in w . Then if $R > r$, the path $p(t^R w)$ starting at 1 ends at distance $|\mu(w)|$ from the left side of Ω , and at level $R + t_{\max}(w)$.*

Proof. If w has length 0 then the statement is true. Suppose for induction the statement is true for all words of length k and let w have length $k+1$.

If $w = ua^{\pm 1}$ then $t_{\max}(w) = t_{\max}(u)$, $\mu(w) = \mu(u)$ and $p(t^R w) = p(t^R u)$ so the statement is true.

If $w = ut^{-1}$ then $t_{\max}(w) = t_{\max}(u)$, $\mu(w) = \mu(u) - 1$. The path $p(t^R u)$ ends at level $R + t_{\max}(u)$ distance $|\mu(u)|$ from the left side of Ω by inductive assumption. By Corollary 6.7 the length of level $R + t_{\max}(u)$ is $R + t_{\max}(u)$, so since $r < R$, the path does not reach the right side of Ω , so appending t^{-1} to the path does not change level, and the distance from the left side is increased by 1, so the statement is true for w since $|\mu(w)| = |\mu(u) - 1| = |\mu(u)| + 1$.

If $w = ut$, then we have two cases. If $t_{\max}(w) = t_{\max}(u) + 1$, then $\mu(w) = 0 = \mu(u)$, so the statement is true since $p(t^R u)$ ends on the left side of Ω and appending a t edge moves it down one level along a vertical edge.

Otherwise $t_{\max}(w) = t_{\max}(u)$. In this case $\mu(u) < 0$, so by inductive assumption $p(t^R u)$ ends at level $R + t_{\max}(u)$ and positive distance $|\mu(u)|$ from the left side of Ω , so appending a t edge does not change the level, and decreases the distance from the left by 1, proving the claim. \square

Proposition 6.10. *If $x \in \text{BS}(m, n)$ has t -exponent sum ρ , then $s(x) = \left(\frac{\text{lcm}(m, n)}{|n|}\right)^\rho$ if $\rho \geq 0$, and $s(x) = \left(\frac{\text{lcm}(m, n)}{|m|}\right)^{|\rho|}$ if $\rho < 0$.*

Proof. We first consider the case $\rho \geq 0$. Take w conjugate to x such that w^k is freely reduced and has no pinches, given by Corollary 6.5. If w has no $t^{\pm 1}$ letters then clearly the closure $\langle a \rangle$ is minimising for w as we are done. If w contains no t^{-1} letters, then $p(w^k)$ is a path in Ω from 1 to $m \left(\frac{\text{lcm}(m, n)}{n}\right)^{\rho^k - 1}$ down the left side ρ^k edges, where ρ is the number of t letters in w . It follows (from Lemma 6.8) that $w^{-k} \langle a \rangle w^k \cap \langle a \rangle = \langle a^m \left(\frac{\text{lcm}(m, n)}{n}\right)^{\rho^k - 1} \rangle$, so by Theorem 2.3

$$\begin{aligned} s(w) &= \lim_{k \rightarrow \infty} [V : V \cap w^{-k} V w^k]^{\frac{1}{k}} \\ &= \lim_{k \rightarrow \infty} \left(|m| \left(\frac{\text{lcm}(m, n)}{|n|}\right)^{\rho^k - 1}\right)^{\frac{1}{k}} \\ &= \left(\frac{l}{|n|}\right)^\rho \end{aligned}$$

where V is the closure of $\langle a \rangle$.

Now suppose w contains both t^{-1} and t letters. Then w is conjugate to a word $u = tvt^{-1}a^\eta$, $\eta \in \mathbb{Z}$. Let N be the number of t^{-1} letters in u . We will compute

the scale of the words $t^{2N}u^kt^{-2N}$ which is conjugate to x , is freely reduced and contains no pinches.

Let t_{\max} be the maximum t -exponent sum of any prefix of u , and write $u = u_1u_2$ where u_1 has t -exponent sum t_{\max} , and u_2 has t -exponent sum $\mu = \rho - t_{\max} \leq 0$ as in Lemma 6.9. Then by Lemma 6.9, the path labeled $p(t^{2N}u)$ starting at 1 ends at distance μ from the left side of Ω , and at level $2N + t_{\max}$. Put $L = 2N + t_{\max}$ and call the endpoint of $p(t^{2N}u)$ the point P_1 .

Now $p(t^{2N}u^2)$ travels along $p(t^{2N}u)$ to P_1 , then along a path labeled $p(u)$. Since u has N t^{-1} letters, $p(t^{2N}u^2)$ also does not reach the right side of Ω .

Since $t_{\max} = \rho + |\mu|$, the path $p(u_1)$ starting at P must reach the left side and either travel along it for at least one t edge (if $\rho > 0$), or if $\rho = 0$ it will stay on level L and end on the left side.

In the case that $\rho = 0$, $p(t^{2n}u^k)$ will end at the point P_1 for all $k \geq 1$. Thus $p(t^{2n}u^kt^{-2N})$ will end at the same point in Ω for all $k \geq 1$, so the index of $u^{-k}\langle a \rangle u^k \cap \langle a \rangle$ is constant as k increases, and by Theorem 2.3 the scale of u is 1.

In the case that $\rho > 0$ and $p(t^{2N}uu_1)$ travels down a level from the point P_1 , it must end on the left side since u_1 has maximal t -exponent sum. Since the t -exponent sum of u_2 is nonpositive, the path $p(u_2)$ from the left side stays at this level, and ends distance $|\mu|$ from the left side once again. Let P_2 be the endpoint of $p(t^{2N}u^2)$. Then the geodesic path from P_1 to P_2 in Ω is $t^\mu t^d t^{-\mu}$ (since the left side is closer than the right at level $2N + t_{\max}$), where d is the difference in levels, and since u is also a path from P_1 to P_2 and has t -exponent sum ρ , we must have $d = \rho$.

Inductively we have that $p(t^{2N}u^k)$ ends at a point P_k at level $L + \rho^{k-1}$ at distance $|\mu|$ from the left side. For k sufficiently large, the point P_k lies further than $2N$ from the right side of Ω , so $p(t^{2N}u^kt^{-2N})$ ends at a point labeled

$$l\left(\frac{l}{n}\right)^{L+\rho^{k-1}-|\mu|-2N} \left(\frac{l}{m}\right)^{|\mu|-1+2N} = l\left(\frac{l}{n}\right)^{t_{\max}+\rho^{k-1}-|\mu|} \left(\frac{l}{m}\right)^{2N+|\mu|-1}.$$

Applying Theorem 2.3 we have

$$\begin{aligned} s(x) &= \lim_{k \rightarrow \infty} [V : V \cap (t^{2N}u^kt^{-2N})^{-1}V(t^{2N}u^kt^{-2N})]^{\frac{1}{k}} \\ &= \lim_{k \rightarrow \infty} \left(l\left(\frac{l}{n}\right)^{t_{\max}+\rho^{k-1}-|\mu|} \left(\frac{l}{m}\right)^{2N+|\mu|-1} \right)^{\frac{1}{k}} \\ &= \left(\frac{l}{n}\right)^\rho \end{aligned}$$

where V is the closure of $\langle a \rangle$.

The case that $\rho < 0$ is treated with the same argument as above, exchanging the left side of Ω for the right, and considering $t^{-2N}u^kt^{2N}$. \square

Putting the results of this section together, and noting that scales of elements of $G_{m,\pm m}$ are 1, we have

Corollary 6.11. *If $x \in BS(m, n)$ for $m, n \neq 0$ has t -exponent sum ρ then $s(x) = \left(\frac{\text{lcm}(m, n)}{|n|}\right)^\rho$ if $\rho \geq 0$ and $\left(\frac{\text{lcm}(m, n)}{|m|}\right)^{|\rho|}$ if $\rho < 0$.*

and so

Theorem 6.12. *The set of scales of elements in $G_{m,n}$ for $m, n \neq 0$ is*

$$\left\{ \left(\frac{\text{lcm}(m, n)}{|n|}\right)^\rho, \left(\frac{\text{lcm}(m, n)}{|m|}\right)^\rho \mid \rho \in \mathbb{N} \right\}.$$

7. THE MODULAR FUNCTION AND FLAT RANK

The *modular function* of a locally compact group G is the homomorphism $\Delta : G \rightarrow \mathbb{R}^+$ which measures how far a left-invariant Haar integral on G is from being right-invariant (see for example [12]).

In [23, 24] the second author proves that in a totally disconnected locally compact group, $\Delta(x) = s(x)/s(x^{-1})$. It follows from Corollary 6.11 that the modular function for elements of $BS(m, n)$ of t -exponent sum ρ is $\left| \frac{m}{n} \right|^\rho$.

Lemma 7.1. *If $x \in G_{m,n}$ then there is some $w \in BS(m, n)$ such that $s(x) = s(w)$ and $s(x^{-1}) = s(w^{-1})$.*

Proof. Let $s(x) = c$ and $s(x^{-1}) = d$. Let ι be the anti-automorphism of $G_{m,n}$ which sends each group element to its inverse. Since s and ι are continuous, the sets $s^{-1}(c)$ and $\iota(s^{-1}(d))$ are open, so their intersection U is open and contains x . Since $BS(m, n)$ is dense in $G_{m,n}$, U contains an element $w \in BS(m, n)$, so $s(w) = c$, and $s(w^{-1}) = d$. \square

It follows that $\Delta(x) = \left| \frac{m}{n} \right|^\rho$ for some $\rho \in \mathbb{Z}$ for all $x \in G_{m,n}$.

Corollary 7.2. *The t -exponent map extends continuously to $G_{m,n}$ and is a group homomorphism.*

Proof. The map $\phi : G_{m,n} \rightarrow \mathbb{Z}$ defined by $\phi(x) = \log_{m/n}(\Delta(x))$ is a well defined group homomorphism, and $\phi|_{BS(m,n)}$ is the t -exponent sum map. \square

Now that each element of $G_{m,n}$ has a well defined notion of t -exponent sum, we can sharpen the main theorem to

Corollary 7.3. *The scale of $x \in G_{m,n}$ is $\left(\frac{\text{lcm}(m,n)}{c} \right)^{|\rho|}$ where ρ is the t -exponent sum of x , $c = |m|$ if $\rho \geq 0$ and $c = |n|$ if $\rho < 0$.*

A subgroup P of a totally disconnected locally compact group is said to be *flat* if some compact open subgroup V is minimizing for all $x \in P$. For example, if $x \in G$ is any element then $\langle x \rangle$ is flat by Proposition 2.1(iii). Let P be flat in G , with V minimizing for each $x \in P$, consider the set $P_1 = \{x \in P \mid s(x) = s(x^{-1}) = 1\}$. It follows from Proposition 2.1(v) that P_1 is a subgroup of P , and by Proposition 2.1(iv) that P_1 is normal. This subgroup is called the *uniscalar subgroup* of P . The second author showed that P modulo its uniscalar subgroup is free abelian of some rank $r \in \mathbb{N} \cup \{\infty\}$ [25]. The number r is called the *flat rank* of P . Define the *flat rank* of a totally disconnected locally compact group G to be the supremum of the flat rank over all flat subgroups of G .

Proposition 7.4. *For $m, n \neq 0$, the flat rank of $G_{m,n}$ is 1 for $|m| \neq |n|$, and 0 for $|m| = |n|$.*

Proof. If $|m| = |n|$, then $G_{m,n}$ has the discrete topology and all elements have scale 1, and so the flat rank of any flat subgroup is 0.

Suppose that $|m| \neq |n|$ and let P be a flat subgroup of $G_{m,n}$. Let E be the kernel of the t -exponent map defined in Corollary 7.2, which is the set of elements x in $G_{m,n}$ with $s(x) = s(x^{-1}) = 1$. Then P/P_1 embeds into $G_{m,n}/E$. Since the t -exponent sum map is surjective, $G_{m,n}/E$ is isomorphic to \mathbb{Z} . Hence P/P_1 embeds

into \mathbb{Z} and the rank of P/P_1 is at most 1. Since $\langle t \rangle$ is a flat subgroup of flat rank 1, the flat rank of $G_{m,n}$ is equal to 1. \square

8. THE LOCAL STRUCTURE OF $G_{m,n}$

In this section we give a more detailed description of the closure of $\langle a \rangle$ in $G_{m,n}$. Using this description and the machinery developed by the second author in [23, 24, 25] the computation of scale becomes much faster. Since we wish to reprove results from the previous two sections, we only assume results from Sections 2 to 5. We do however continue to abuse notation and identify elements and subsets of $\text{BS}(m, n)$ with their images under the embedding π .

Recall that an inverse (or projective) system of groups and homomorphisms is a family of groups $(A_i)_{i \in I}$ indexed by a directed poset (I, \leq) , and a family of homomorphisms $f_{ij} : A_j \rightarrow A_i$ for all $i \leq j$ with the following properties:

- (1) f_{ii} is the identity on A_i ,
- (2) $f_{ik} = f_{ij} \circ f_{jk}$ for all $i \leq j \leq k$.

The *inverse limit* of the inverse system $((A_i)_{i \in I}, (f_{ij})_{i \leq j \in I})$ is the subgroup

$$\varprojlim_{i \in I} A_i = \left\{ (a_i)_{i \in I} \in \prod_{i \in I} A_i \mid a_i = f_{ij}(a_j) \text{ for all } i \leq j \text{ in } I \right\}$$

of the direct product $\prod_{i \in I} A_i$.

An example of an inverse limit is the additive group of p -adic integers – take $A_i = \mathbb{Z}/p^i\mathbb{Z}$ and f_{ij} the remainder map modulo p^i .

If each A_i is finite, the inverse limit is called a *profinite group*. It is a consequence of Tychonov's theorem that profinite groups are compact. Conversely, every compact totally disconnected group is isomorphic to a profinite group, [17, Theorem 1.1.12]. If a topological group contains a dense cyclic subgroup, it is said to be *monothetic* [12, Definition 9.2]. The closure of $\langle a \rangle$ in $G_{m,n}$ is compact, totally disconnected and monothetic. Hence, by [12, Theorem 25.16],

$$(4) \quad \overline{\langle a \rangle} \cong \prod_{p \in \Gamma} Z_p,$$

where Γ is the set of prime numbers and Z_p is isomorphic to the p -adic integers, \mathbb{Z}_p , or a (finite) quotient of this group.

The first aim of this section is determine the groups Z_p by examining the details of the construction of $G/\!/H$ given in Section 5 for this particular case. As seen in Equation (3), that construction identifies $\overline{\langle a \rangle}$ with a subgroup of a product of finite permutation groups which, since $\langle a \rangle$ is cyclic, act by cyclicly permuting the cosets in $\langle a \rangle \cdot g \langle a \rangle$ for each g in $\text{BS}(m, n)$. Therefore Equation 3 implies that

$$\overline{\langle a \rangle} \leq \prod \mathbb{Z}/d\mathbb{Z},$$

where the product is indexed by the cosets $g \langle a \rangle$ in $\text{BS}(m, n)/\langle a \rangle$, and d is the order of the corresponding orbit $\langle a \rangle \cdot g \langle a \rangle$.

Proposition 8.1. *The compact open subgroup $\overline{\langle a \rangle}$ has an open subgroup V such that*

$$V \cong \prod \{\mathbb{Z}_p \mid p \text{ is a prime divisor of } \text{lcm}(m, n)/m \text{ or } \text{lcm}(m, n)/n\}.$$

The quotient $\overline{\langle a \rangle}/V$ is a finite cyclic group with order dividing $\text{gcd}(m, n)$.

Proof. Let $w\langle a \rangle \neq \langle a \rangle$ be a coset and write $w = a^{\eta_1} t^{\epsilon_1} a^{\eta_2} t^{\epsilon_2} \dots a^{\eta_k} t^{\epsilon_k}$ with $\epsilon_i = \pm 1$ and $k > 0$. To determine the size of the $\langle a \rangle$ -orbit of $w\langle a \rangle$, consider d with $a^d w\langle a \rangle = w\langle a \rangle$. It will be useful to compute e such that $a^d w = w a^e$.

The defining relation for $\text{BS}(m, n)$ implies that if $a^d w = w a^e$ then

$$a^d w = a^{\eta_1} t^{\epsilon_1} a^{\eta_2} t^{\epsilon_2} \dots a^{\eta_k} t^{\epsilon_k},$$

where $d' = d \frac{m}{n}$ or $d \frac{n}{m}$ depending on whether ϵ_1 is equal to 1 or -1 . Continuing to push d' past t^{ϵ_i} for $i = 2, \dots, k$, we find that

$$(5) \quad e = d \left(\frac{m}{n} \right)^\rho = d \left(\frac{\frac{\text{lcm}(m, n)}{n}}{\frac{\text{lcm}(m, n)}{m}} \right)^\rho,$$

where ρ is the t -exponent of w . Indeed, this calculation goes through provided that the $\text{gcd}(m, n)$ and sufficiently high powers of $\frac{\text{lcm}(m, n)}{n}$ and $\frac{\text{lcm}(m, n)}{m}$ divide d . Hence there are $\langle a \rangle$ -orbits with orders

$$(6) \quad d = \text{gcd}(m, n) \left(\frac{\text{lcm}(m, n)}{m} \right)^r \left(\frac{\text{lcm}(m, n)}{n} \right)^s$$

for some $r, s \in \mathbb{N}$. Since $\frac{\text{lcm}(m, n)}{n}$ and $\frac{\text{lcm}(m, n)}{m}$ are relatively prime and the t -exponent of w may take any integer value, it follows from (5) that r and s may take arbitrarily high values. Therefore, if p is a prime divisor of $\frac{\text{lcm}(m, n)}{n}$ or $\frac{\text{lcm}(m, n)}{m}$, then Z_p in (4) is isomorphic to \mathbb{Z}_p .

To see that all orbits have the orders given in (6), choose a coset representative w that is freely reduced and has no pinches. Then the equation $w^{-1} a^d w = a^e$ implies that $t^{-\epsilon_1} a^d t^{\epsilon_1}$ is a pinch. Hence either m or n divides d , depending on whether ϵ_1 equals -1 or 1 . Reduction of pinches continues for all ϵ_i provided that sufficient powers of $\frac{\text{lcm}(m, n)}{m}$ and $\frac{\text{lcm}(m, n)}{n}$ divide d , thus showing that all orbits have orders as given in (6), as claimed. This implies that, if p does not divide $\text{gcd}(m, n)$, $\frac{\text{lcm}(m, n)}{n}$ or $\frac{\text{lcm}(m, n)}{m}$, then Z_p in (4) is trivial, and that, if p divides $\text{gcd}(m, n)$ but not $\frac{\text{lcm}(m, n)}{n}$ or $\frac{\text{lcm}(m, n)}{m}$, then Z_p is finite cyclic with order dividing $\text{gcd}(m, n)$. \square

It follows from Proposition 8.1 that $G_{m, n}$ is a closed subgroup of a product of p -adic Lie groups with p belonging to

$$\mathfrak{p} := \left\{ p \mid p \text{ is a prime divisor of } \frac{\text{lcm}(m, n)}{n} \text{ or } \frac{\text{lcm}(m, n)}{m} \right\}.$$

Hence $G_{m, n}$ belongs to the class $\mathbb{A}_{\mathfrak{p}}$ of groups defined in [10]. It is shown in [10] that Hausdorff groups in the variety generated by p -adic Lie groups, for $p \in \mathfrak{p}$, may be approximated by groups in $\mathbb{A}_{\mathfrak{p}}$.

When the t -exponent of the coset representative w equals 0, the above calculations yield further information.

Corollary 8.2. *If the t -exponent of $w \in \text{BS}(m, n)$ is equal to 0, then there is $d > 0$ such that w centralizes $\langle a^d \rangle$ and (the image under π of) w centralizes $\overline{\langle a^d \rangle}$.*

Using this description we now revisit Corollary 7.2 where the notion of t -exponent sum was extended to $G_{m, n}$. Let $x \in G_{m, n}$.

Recall that every locally compact group admits a (left-invariant) Haar measure, which is finitely additive, invariant under left translation by group elements, and unique up to rescaling. Let μ be such a measure on $G_{m, n}$, scaled so that $\mu(\overline{\langle a \rangle}) = 1$.

Then $\mu(\overline{\langle a^j \rangle}) = \frac{1}{j}$ for any positive integer j since the measure is translation invariant (so all cosets have the same measure) and finitely additive. The modular function $\Delta : G \rightarrow \mathbb{R}^+$ is defined as follows. For any $g \in G$ and compact set $A \subseteq G$, $\mu(Ag) = \Delta(g)\mu(A)$. In the case that G is totally disconnected, as mentioned in Section 7, the second author showed that the modular function always takes rational values. Since Δ is a continuous map from $G_{m,n}$ to \mathbb{Q}^+ , and $\text{BS}(m,n)$ is dense in $G_{m,n}$, there exists some \tilde{x} in $\text{BS}(m,n)$ such that $\Delta(x) = \Delta(\tilde{x})$. Then $\tilde{x} = t^\rho w$ where w is a word of zero t -exponent sum. Since Δ is a homomorphism, $\Delta(\tilde{x}) = (\Delta(t))^\rho \Delta(w)$.

By Corollary 8.2 the word w centralizes a compact open subgroup, and so $\Delta(w) = 1$. Since μ is left invariant,

$$\mu(t^{-1}\overline{\langle a^n \rangle}) = \mu(\overline{\langle a^n \rangle}) = \frac{1}{|n|}.$$

Translating the compact open subgroup $t^{-1}\overline{\langle a^n \rangle}$ by t on the right we have

$$\mu(t^{-1}\overline{\langle a^n \rangle}t) = \mu(\overline{\langle a^m \rangle}) = \frac{1}{|m|}.$$

It follows that

$$\frac{1}{|n|} = \mu(t^{-1}\overline{\langle a^n \rangle}) = \Delta(t)\mu(t^{-1}\overline{\langle a^n \rangle}t) = \Delta(t)\frac{1}{|m|}$$

and so $\Delta(t) = \left|\frac{m}{n}\right|$.

Putting this together we have $\Delta(x) = \Delta(\tilde{x}) = (\Delta(t))^\rho \Delta(w) = \left|\frac{m}{n}\right|^\rho$, and so we may define the function $\phi : G_{m,n} \rightarrow \mathbb{Z}$ as in the proof of Corollary 7.2 which extends the t -exponent sum to all of $G_{m,n}$.

The *quasi-centre*, $\text{QZ}(G)$, of the topological group G is defined in [4] to be set of elements of G that centralize an open subgroup of G . Since the intersection of any two open subgroups is an open subgroup and the set of open subgroups is invariant under automorphisms, $\text{QZ}(G)$ is a characteristic subgroup of G . It is typically not closed, but when G has an open abelian subgroup it is open.

Corollary 8.3. *The quasi-centre of $G_{m,n}$ is equal to $\ker \Delta$.*

Proof. Since $\overline{\langle a \rangle}$ is abelian, it is contained in $\text{QZ}(G_{m,n})$. The smallest subgroup of $G_{m,n}$ containing $\overline{\langle a \rangle}$ and the image under π of all elements of $\text{BS}(m,n)$ having t -exponent 0 is $\ker \Delta$. Hence $\text{QZ}(G_{m,n})$ contains this subgroup. The reverse inclusion holds because every element of the quasi-centre is unimodular. \square

Computation of the scale of $x \in G_{m,n}$ may now be carried out more directly with the aid of the relevant structure theory as described in [23]. Recall in particular that the compact, open subgroup, V is *tidy* for x if, setting $V_+ = \bigcap_{k \geq 0} x^k V x^{-k}$ and $V_- = \bigcap_{k \geq 0} x^{-k} V x^k$, we have

$$V = V_+ V_- \quad \text{and} \quad \bigcup_{k \geq 0} x^k V_+ x^{-k} \text{ is closed.}$$

When V is tidy for x we have $s(x) = [xV_+x^{-1} : V_+]$ (and so tidy subgroups are minimising).

Proposition 8.4. *Let $x \in G_{m,n}$ and denote the t -exponent of x by ρ . Then there is a compact, open subgroup, V , tidy for x with $V = V_+V_-$ where*

$$V_+ \cong \prod \{\mathbb{Z}_p \mid p \text{ is a prime divisor of } \text{lcm}(m,n)/n\}$$

$$\text{and } V_- \cong \prod \{\mathbb{Z}_p \mid p \text{ is a prime divisor of } \text{lcm}(m,n)/m\}$$

if ρ is positive and vice versa if ρ is negative. The scale of x is

$$s(x) = \begin{cases} \left(\frac{\text{lcm}(m,n)}{|n|}\right)^\rho, & \text{if } \rho \geq 0 \\ \left(\frac{\text{lcm}(m,n)}{|m|}\right)^{-\rho}, & \text{if } \rho \leq 0 \end{cases}.$$

Proof. Write $x = t^\rho(t^{-\rho}x) = t^\rho w$ where $w \in G_{m,n}$ is in $\ker \Delta$. By Corollary 8.3, there is an open subgroup $V \leq \overline{\langle a \rangle}$ that is centralised by w . It follows from Proposition 8.1 and [23, Lemma 1] that we may, by passing to a subgroup if necessary, suppose that

$$(7) \quad V \cong \prod \{\mathbb{Z}_p \mid p \text{ is a prime divisor of } \text{lcm}(m,n)/m \text{ or } \text{lcm}(m,n)/n\}$$

and $V = V_+V_-$. Note that, since V is an open subgroup of $\overline{\langle a \rangle}$, there is $d > 0$ such that $V = \overline{\langle a^d \rangle}$ and it may also be supposed that m, n and $(n/m)^\rho$ divide d .

Since w centralises V , we have $xVx^{-1} = t^\rho V t^{-\rho}$. Hence

$$xVx^{-1} = \overline{\langle t^\rho a^d t^{-\rho} \rangle} = \overline{\langle a^e \rangle} \text{ where } e = d(\frac{n}{m})^\rho.$$

Then under the isomorphism (7) of V with the product of groups of p -adic integers, which converts multiplicative notation to additive, conjugation by x multiplies each factor \mathbb{Z}_p by $(\frac{m}{n})^\rho$. It follows that, if ρ is positive, conjugation by x expands the factor \mathbb{Z}_p if p divides $\frac{\text{lcm}(m,n)}{n}$ and contracts it if p divides $\frac{\text{lcm}(m,n)}{m}$, and vice versa if ρ is negative. Therefore V_+ and V_- are as claimed. It further follows that $\bigcap_{k \in \mathbb{Z}} x^k V x^{-k}$ is trivial whence, by [3, Lemma 3.31(3)], V is tidy for x . Therefore $s(x) = [xV_+x^{-1} : V_+]$ and has the claimed values. \square

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